Lecture 33

Consider again the line integral

$$\int_C P\,dx + Q\,dy + R\,dz.$$

Let's mess around with the integrand:

$$\int_C P\,dx + Q\,dy + R\,dz = \int_C \langle P, Q, R\rangle \cdot \langle dx, dy, dz\rangle$$

What type of object is $\langle P, Q, R\rangle$? (We'll discuss $\langle dx, dy, dz\rangle$ later.) This leads us to:

16.1 - Vector Fields

Def: A vector field on $\mathbb{R}^2$ is a function $\vec{F}$ that assigns to each point (x,y) in its domain (a subset of $\mathbb{R}^2$) a 2-D vector $\vec{F}(x,y)$. We can also write $\vec{F}$ in terms of its component functions

$$\vec{F}(x,y) = \langle P(x,y), Q(x,y)\rangle = P(x,y)\,\hat{i} + Q(x,y)\,\hat{j}$$
$$= \langle P, Q\rangle \quad = P\hat{i} + Q\hat{j}$$

Of course, we can define vector fields on $\mathbb{R}^3$ as well in a similar fashion:

$$\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle = \langle P, Q, R\rangle.$$

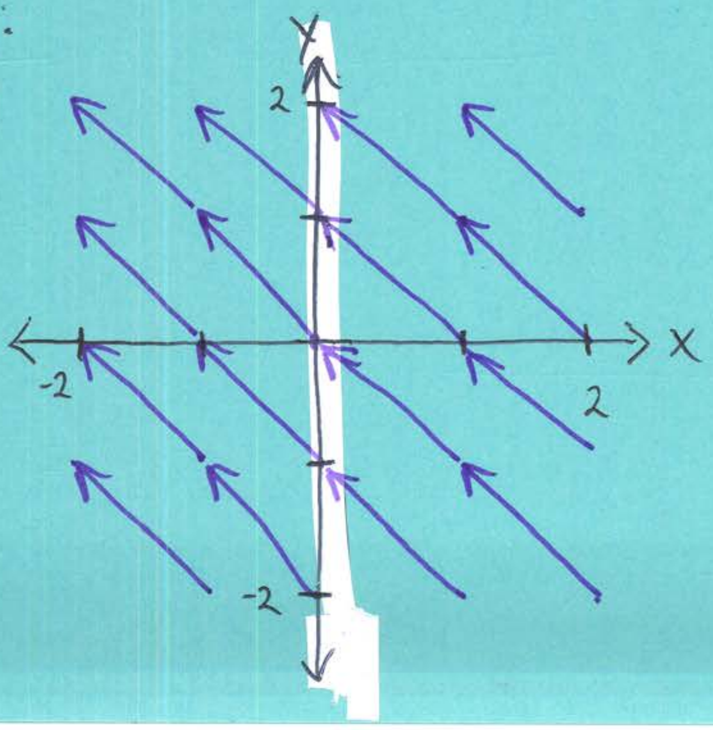
So, what do vector fields look like? Well, at each point (x,y) in its domain, it gives us the vector $\langle P(x,y), Q(x,y)\rangle$, so we draw that vector starting at the point (x,y) (i.e., (x,y) is the tail of the vector $\vec{F}(x,y)$).

Ex: Sketch the vector field

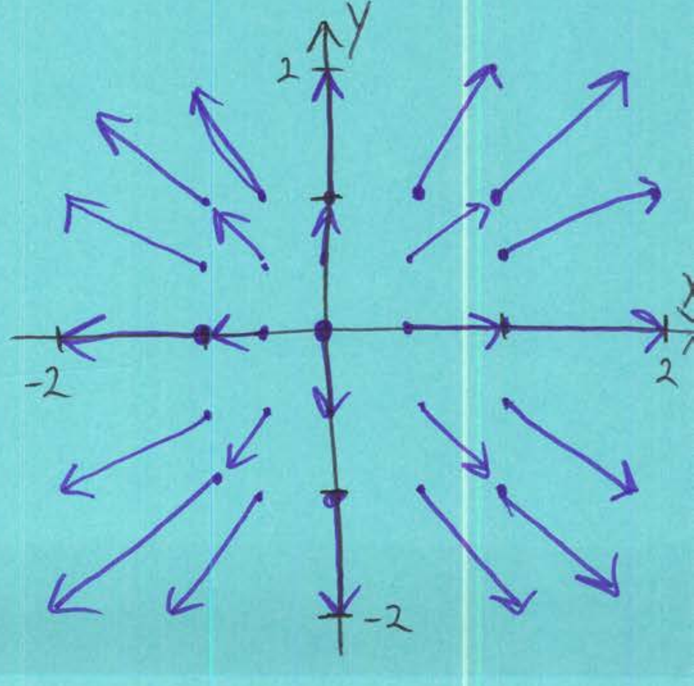
a) $\vec{F}(x,y) = \langle -1, 1\rangle$, b) $\vec{F}(x,y) = \langle x,y\rangle$, c) $\vec{F}(x,y) = \langle 2y, 0\rangle$

Sol:

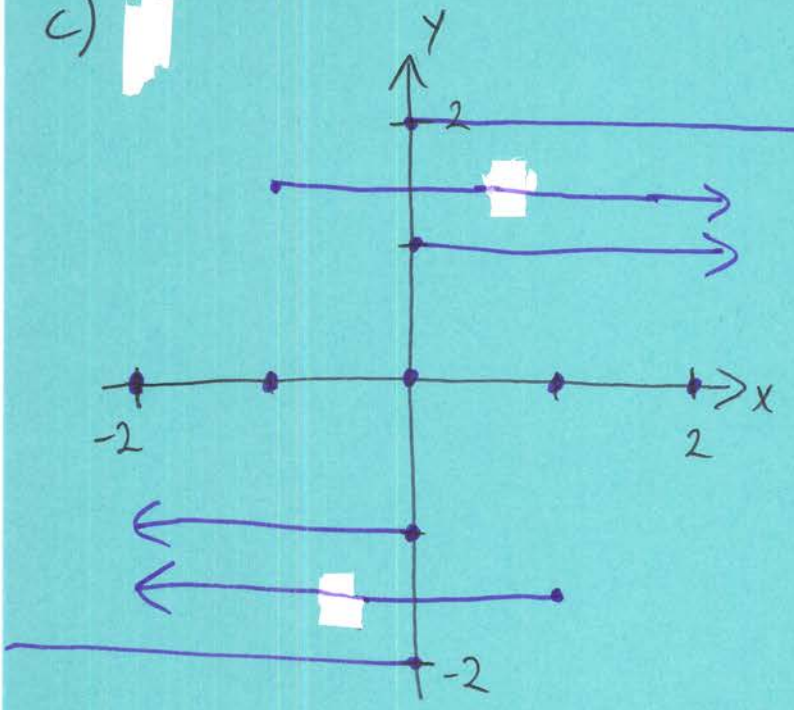
a)



b)



c)



(The mathematica code has a MUCH better picture for this one.)

$\diamond$

Vector fields usually look better if you scale all the vectors down in the plot.

Let's see some 3-D vector fields (plotted in mathematica).

<u>Ex</u>: Plot

a) $\vec{F}(x,y,z) = \langle x, 0, 0 \rangle$, b) $\vec{F}(x,y,z) = \langle y, -x, z \rangle$

c) $\vec{F}(x,y,z) = \langle 2, 1, -3 \rangle$, d) $\vec{F}(x,y,z) = \langle -x, -y, -z \rangle$

Some practical examples of vector fields:

- Suppose that there is a body of mass M at $(0,0,0)$. The gravitational force exerted on a body of mass m with position vector $\vec{r} = \langle x, y, z \rangle$ is
$$\vec{F}(x,y,z) = \vec{F}(\vec{r}) = -\frac{mMG}{|\vec{r}|^2} \frac{\vec{r}}{|\vec{r}|} = -\frac{mMG}{|\vec{r}|^3} \vec{r}$$

- Suppose there is a charge Q at the origin. The force it exerts on a charge q with position vector $\vec{r} = \langle x, y, z \rangle$ is

$$\vec{E}(\vec{r}) = \frac{kqQ}{|\vec{r}|^2} \frac{\vec{r}}{|\vec{r}|} = \frac{kqQ}{|\vec{r}|^3} \vec{r}$$

- Another example is a magnetic field

Gradient Vector Fields

A gradient vector field is a vector field of the form $\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$ or
$\nabla f(x,y,z) = \langle f_x, f_y, f_z \rangle$.

The vector field $\langle -1, 1\rangle$ is a gradient vector field where $f = y - x$ and the vector field $\langle -x, -y, -z\rangle$ is a gradient vector field where $f = -\frac{1}{2}(x^2+y^2+z^2)$.

A vector field $\vec{F}$ which is a gradient vector field, i.e., $\vec{F} = \nabla f$, is called <u>conservative</u>. The function f is called a <u>potential function</u> for $\vec{F}$.

<u>16.2</u> - Line Integral (Part II)

Back to $\int_C P\,dx + Q\,dy + R\,dz$.

We changed $\int_C P\,dx + Q\,dy + R\,dz$ to $\int_C \langle P, Q, R\rangle \cdot \langle dx, dy, dz\rangle$.

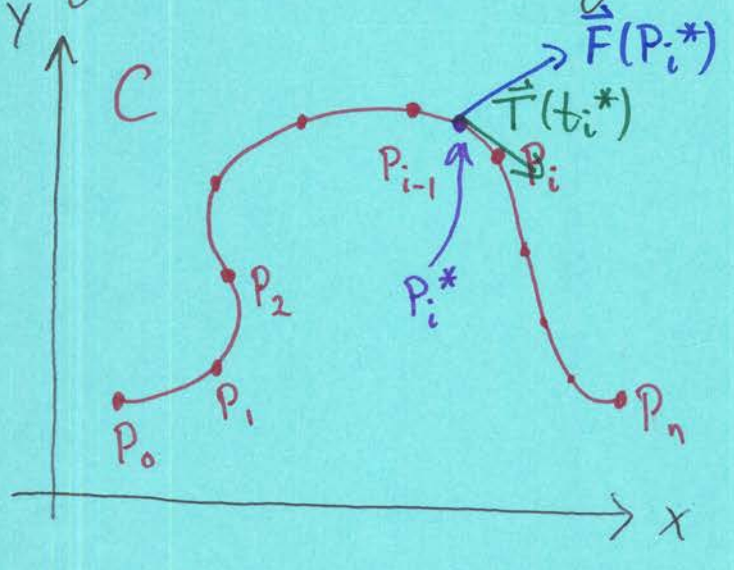
If we let $\vec{r} = \langle x, y, z\rangle$, then $d\vec{r} = \langle dx, dy, dz\rangle$, so if $\vec{F} = \langle P, Q, R\rangle$ this integral takes the form

$$\int_C \vec{F} \cdot d\vec{r}.$$

What are these good for? We can use them for <u>work</u>.

Let's say we have a particle moving along a path C in the presence of a force field $\vec{F}$.

We know that the work done by a constant force $\vec{F}$ in moving along a straight line is $W = \vec{F} \cdot \vec{d}$, where $\vec{d}$ is the displacement vector. When C is not a straight line, we compute the work done by approximating it over small segments of the curve:



C is parametrized by $\vec{r}(t)$, $\vec{T}(t)$ is the unit tangent vector of $\vec{r}(t)$ which describes which direction you move along the curve, and the others are as before. We approx. the work done by $\vec{F}$ from P_{i-1} to P_i by

$$W_i = \vec{F}(P_i^*) \cdot \vec{T}(t_i^*) \Delta s_i$$

Then, we have

$$\boxed{W = \lim_{n\to\infty} \sum_{i=1}^{n} W_i = \lim_{n\to\infty} \sum_{i=1}^{n} \vec{F}(P_i^*)\cdot\vec{T}(t_i^*)\Delta s_i = \int_C \vec{F}\cdot\vec{T}\, ds}$$

To compute W, we use the following

$$W = \int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot\vec{T}\, ds = \int_a^b \vec{F}(\vec{r}(t))\cdot \frac{\vec{r}\,'(t)}{\cancel{|\vec{r}\,'(t)|}} \cancel{|\vec{r}\,'(t)|}\, dt$$

$$= \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\, dt$$

In summary, the <u>line integral of $\vec{F}$ along C</u> is given by $\displaystyle\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\, dt$ where $\vec{r}(t)$, $a\le t\le b$, is a parametrization of C.

<u>Ex</u>: Compute $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = \langle xy, 3y^2\rangle$ and C is parametrized by: $\vec{r}(t) = \langle 11t^4, t^3\rangle$, $0\le t\le 1$.

<u>Sol</u>: $\vec{r}\,'(t) = \langle 44t^3, 3t^2\rangle$, $\vec{F}(\vec{r}(t)) = \langle 11t^7, 3t^6\rangle$, so:

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\, dt = \int_0^1 (484t^{10} + 9t^8)\, dt = (44t^{11} + t^9)\Big|_0^1 = 45$$

$\diamond$

Some other uses of line integrals:

- **Ampère's Law**: Suppose we have a wire carrying a current I and that the wire forms a closed loop C. If $\vec{B}$ is the magnetic field around the wire, then $$I = \frac{1}{\mu_0} \oint_C \vec{B} \cdot d\vec{r}.$$

- **Kirchhoff's Law**: Under sufficiently ideal conditions the sum of the voltage drops around a closed circuit is zero. In terms of line integrals: $$\oint_C \vec{E} \cdot d\vec{r} = 0.$$

- **Faraday's Law**: The electromotive force (EMF) $\mathcal{E}$ induced around a closed loop C is $$\mathcal{E} = \frac{1}{q} \oint_C \vec{F} \cdot d\vec{r} = \oint_C \left(\vec{E} + \vec{v} \times \vec{B}\right) \cdot d\vec{r}$$ where $\vec{F}$ is the Lorentz force on a point charge moving with velocity $\vec{v}$.